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# RESULTS ON MEASURES OF IRREDUCIBILITY AND FULL INDECOMPOSABILITY(1)

BY

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ABSTRACT. This paper develops a notion of $k$th measure of irreducibility and $k$th measure of full indecomposability. The combinatorial properties of these notions, as well as relationships between these notions, are explored. The results are then used in converting results on positive matrices into results on nonnegative matrices.

Let $n \geqslant 2$ be an integer, $N = \{1, 2, \cdots, n\}$. All matrices considered shall be $n \times n$ and nonnegative. Let

$$u_k(A) = \min_{R \cap C = \emptyset; |R| + |C| = n-k} \left( \max_{i \in R; j \in C} a_{ij} \right),$$

$k = 0, 1, \cdots, n-2$, and $R, C \subseteq N$ with $|S|$ denoting the number of elements in set $S$. Verbally, $u_k(A)$ is computed as follows. Choose the largest entry in each $|R| \times |C|$ submatrix of $A$ not containing an entry from the main diagonal, where $|R| + |C| = n - k$. $u_k(A)$ is then the smallest of all these numbers. For example, if

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 1 & 3 \\ 1 & 2 & 2 \end{pmatrix},$$

then $u_0(A) = 1, u_1(A) = 0$. Note that $u_k(A)$ is independent of the main diagonal entries, i.e. $u_k(A) = u_k(A + D)$ for any diagonal matrix $D$.

Further, set

$$U_k(A) = \min_{|R| + |C| = n-k} \left( \max_{i \in R; j \in C} a_{ij} \right),$$

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$k = 0, 1, \cdots, n-2$. Verbally, $U_k(A)$ is computed as follows. Choose the largest entry in each $|R| \times |C|$ submatrix of $A$ where $|R| + |C| = n - k$. $U_k(A)$ is then the smallest of all these numbers. For example, if

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 1 & 3 \\ 1 & 2 & 2 \end{pmatrix},$$

then $U_0(A) = 1, U_1(A) = 0$.

We call $u_k$ the $k$th measure of $A$'s irreducibility, and $U_k$ the $k$th measure of $A$'s full indecomposability. It is easily seen that $u_k(A) = u_k(PAP^t)$ for any permutation matrix $P$, hence $u_k$ is a $P$-invariant of $A$. Similarly, as $U_k(A) = U_k(PAQ)$, $U_k$ is a $PQ$-invariant of $A$.

The following properties are useful in reading the manuscript.

(1) $u_0(A) > 0 \Leftrightarrow A$ is irreducible.

(2) $U_0(A) > 0 \Leftrightarrow A$ is fully indecomposable.

(3) $u_{n-2}(A) = \min_{i \neq j} a_{ij}$, $U_{n-2}(A) = \min_{i,j} a_{ij}$.

(4) $u_k(A) \geqslant U_k(A)$ for all $k$, $0 \leqslant k \leqslant n-2$.

(5) $u_k(A) \geqslant u_{k+1}(A)$ for all $k$, $0 \leqslant k \leqslant n-3$.

(6) $U_k(A) \geqslant U_{k+1}(A)$ for all $k$, $0 \leqslant k \leqslant n-3$.

(7) Let $\hat{A}$ denote a principal submatrix of $A$ and $n_{\hat{A}}$ its order. Then

$$u_k(A) = \min_{n_{\hat{A}} = n-k} u_0(\hat{A}), \qquad U_k(A) = \min_{n_{\hat{A}} = n-k} U_0(\hat{A}).$$

The above properties then indicate that $u_k$ and $U_k$ "in some sense" measure the strength of $A$'s being irreducible and fully indecomposable, respectively. Alternately, if $A$ is stochastic we may say that $u_k$ and $U_k$ measure the distribution of the entries of $A$.

**Initial results.** A relationship between $u_0$ and $U_0$ is established in [3, p. 33] by showing that

$$u_0(A) > 0 \Leftrightarrow U_0(A + D) > 0,$$

where $D$ is any diagonal matrix with positive main diagonal.

This may be generalized, by the same argument, to the following statement.

LEMMA 1. $u_k(A) > 0 \Leftrightarrow U_k(A + D) > 0$, *where* $D$ *is any diagonal matrix with positive main diagonal.*

Let $\lambda_k = \{A | u_k(A) > 0\}$ and $\Lambda_k = \{A | U_k(A) > 0\}$. We shall say that $A$ is $\min u_k$ if $u_k(A) > 0$, and for each $a_{ij} > 0, u_k(A - a_{ij}E_{ij}) = 0$, where

$E_{ij}$ is the (0, 1)-matrix with a 1 only in its $ij$th position. Min $U_k$ is defined similarly. In the literature, the minimal members are known as follows.

(1) Minimal members of $\lambda_0$ are called "nearly reducible" [4] or in the language of graph theory, "minimally connected".

(2) Minimal members of $\Lambda_0$ are called "nearly decomposable" [12].

Concerning matrices with min $U_k$ or min $u_k$ we have the following.

LEMMA 2. *Suppose* $U_k(A) > 0$. *Then there is a matrix* $B = (b_{ij})$ *with* $b_{ij} \leqslant a_{ij}$ *for each* $i, j \in N$ *so that*

(1) $U_k(B) = U_k(A)$, *and*

(2) $B$ *is* min $U_k$.

PROOF. Replace each $a_{ij} < U_k(A)$ in $A$ by 0 yielding $\bar{A} = (\bar{a}_{ij})$. Note that $U_k(\bar{A}) = U_k(A)$. Now there are sets $R_0$, $C_0$ in $N$ so that $|R_0| + |C_0| = n - k$, and if $\bar{a}_{ij} > 0, i \in R_0$ and $j \in C_0$, then $\bar{a}_{ij} = U_k(\bar{A})$. If $\bar{A}$ is not min $U_k$ there is some $\bar{a}_{i_1j_1}$ in $\bar{A}$ so that $U_k(\bar{A} - \bar{a}_{i_1j_1}E_{i_1j_1}) > 0$ and hence

$$U_k(\bar{A} - \bar{a}_{i_1j_1}E_{i_1j_1}) = U_k(\bar{A}) = U_k(A).$$

As $\bar{A} - \bar{a}_{i_1j_1}E_{i_1j_1}$ has one more 0 than does $\bar{A}$, we may obtain inductively $\bar{a}_{i_2j_2}, \bar{a}_{i_3j_3}, \cdots, \bar{a}_{i_tj_t}$ so that

$$U_k(\bar{A} - \bar{a}_{i_1j_1}E_{i_1j_1} - \bar{a}_{i_2j_2}E_{i_2j_2} - \cdots - \bar{a}_{i_tj_t}E_{i_tj_t}) = U_k(\bar{A}) = U_k(A)$$

and

$$\bar{A} - \bar{a}_{i_1j_1}E_{i_1j_1} - \bar{a}_{i_2j_2}E_{i_2j_2} - \cdots - \bar{a}_{i_tj_t}E_{i_tj_t} = B \quad \text{is min } U_k.$$

Similarly for $u_k$ we have the following.

LEMMA 3. *Suppose* $u_k(A) > 0$. *Then there is a matrix* $B = (b_{ij})$ *with* $b_{ij} \leqslant a_{ij}$ *for each* $i, j \in N$ *so that*

(1) $u_k(B) = u_k(A)$ *and*

(2) $B$ *is* min $u_k$.

Explicit canonical forms for matrices with min $U_0$ and min $u_0$ may be found in [4] and [12]. The structure of matrices with min $U_k$ or min $u_k$ for $k > 0$ has yet to receive attention, although Richard Brualdi and Hazel Perfect [2] have considered the diagonal structure of matrices $A$ such that $U_k(A) > 0$. An interesting characterization which they obtain states that $U_k(A) > 0$ if and only if $A$ is fully indecomposable and each $k + 1$ positive entries, taken from different rows and columns, lie on a common positive diagonal [2, p. 389].

Conforming to Lewin's notation [7], if $x = (x_1, x_2, \cdots, x_n)^t \geqslant 0$,

$I_+(x) = \{i|x_i > 0\}$ and $I_0(x) = \{i|x_i = 0\}$. In [7, p. 755] it is shown that $A$ is fully indecomposable if and only if $|I_0(Ax)| < |I_0(x)|$, i.e. $|I_+(Ax)| > |I_+(x)|$, where $|I_+(x)| > 0$ and $|I_0(x)| > 0$. This result is then utilized to show that if $A$ and $B$ are fully indecomposable then so is $AB$. Further, the result is used to show the well-known fact that if $A$ is fully indecomposable then $A^{n-1} > 0$. We generalize Lewin's work by considering related questions concerning $U_k$.

LEMMA 4. *The following statements are equivalent:*

(1) $U_k(A) > 0$.

(2) $|I_+(Ax)| \geqslant \min\{n, |I_+(x)| + k + 1\}$ *for all* $x \geqslant 0$ *with* $|I_+(x)| > 0$.

PROOF. For (1) $\Rightarrow$ (2) set $|I_+(x)| = s$. Pick $P$, a permutation matrix, so that $Px = y$ with $y_1 > 0, y_2 > 0, \cdots, y_s > 0$. Pick $Q$, a permutation matrix, so that

$$QAP^t = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}$$

with $A_1 r \times s$ having no rows of zeros and $A_3 = 0$. (If $r = n$, $A_3$ and consequently $A_4$ are missing.) Now as $U_k(QAP^t) = U_k(A) > 0$, it follows that if $A_3$ appears, as $s + (n - r) \leqslant n - k - 1, r \geqslant s + k + 1$. Hence $(QAP^t)(Px) = (QAP^t)y = z$ is such that $z_1 > 0, z_2 > 0, \cdots, z_{s+k+1} > 0$ if $s + k + 1 \leqslant n$ and $z_1 > 0, z_2 > 0, \cdots, z_n > 0$ otherwise. Therefore $|I_+(Ax)| \geqslant \min\{n, |I_+(x)| + k + 1\}$.

For (2) $\Rightarrow$ (1) suppose $U_k(A) = 0$. Then there are rows $R_0$, columns $C_0$ with $|R_0| + |C_0| = n - k$ so that for $i \in R_0, j \in C_0$ we have $a_{ij} = 0$. Pick permutation matrices $P$ and $Q$ so that

$$QAP = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix},$$

where $A_3$ is $|R_0| \times |C_0|$ having entries in the rows $R_0$ and columns $C_0$ of $A$. Now pick $x$ so that $x_1 > 0, x_2 > 0, \cdots, x_{|C_0|} > 0, x_{|C_0|+1} = 0, \cdots, x_n = 0$. Consider $(QAP)x = y$. Now $y_{n-|R_0|+1} = 0, \cdots, y_n = 0$. Therefore

$$|I_+(QAPx)| = |I_+(APx)| \leqslant n - |R_0| = |C_0| + k = |I_+(x)| + k,$$

a contradiction. Hence (2) $\Rightarrow$ (1).

LEMMA 5. *The following statements are equivalent:*

(1) $U_k(A) > 0$.

(2) $|I_0(Ax)| \leqslant \max\{0, |I_0(x)| - k - 1\}$ *for all* $x \geqslant 0$.

PROOF. Note that if $y \geqslant 0, |I_+(y)| + |I_0(y)| = n$. The formula allows the conversion of Lemma 4 into Lemma 5.

Direct applications of these lemmas yield the following corollaries. For notational convenience we define $U_k = U_{n-2}, u_k = u_{n-2}$ if $k \geqslant n-2$. Recall that $U_{n-2}(A) = \min_{i,j} a_{ij}$ and $u_{n-2}(A) = \min_{i \neq j} a_{ij}$.

COROLLARY 1. *If* $U_r(A) > 0, U_s(B) > 0$ *then* $U_{r+s+1}(AB) \geqslant U_r(A) U_s(B)$.

PROOF. Suppose $x \geqslant 0$ and $|I_+(x)| > 0$. Note that $|I_+(ABx)| \geqslant |I_+(Bx)|$. If $|I_+(ABx)| < n$ then

$$|I_+(ABx)| \geqslant |I_+(Bx)| + r + 1 = |I_+(x)| + r + s + 2.$$

Hence it follows from Lemma 4 that $U_{r+s+1}(AB) > 0$. Now, by Lemma 2, take $\bar{A}$ so that $U_r(\bar{A}) = U_r(A)$ and $\bar{A}$ is minimal $U_r$. Select $\bar{B}$ similarly for $B$. By the above argument $U_{r+s+1}(\bar{A}\bar{B}) > 0$. By definition $U_{r+s+1}(\bar{A}\bar{B}) \geqslant U_r(\bar{A}) U_s(\bar{B})$. Finally

$$U_{r+s+1}(AB) \geqslant U_{r+s+1}(\bar{A}\bar{B}) \geqslant U_r(\bar{A})\, U_s(\bar{B}) = U_r(A)\, U_s(B).$$

Utilizing the same techniques we may deduce the following corollaries.

COROLLARY 2. *If* $k_1 \geqslant 0, k_2 \geqslant 0, \cdots, k_s \geqslant 0$ *and* $U_{k_1}(A_1) > 0$, $U_{k_2}(A_2) > 0, \cdots, U_{k_s}(A_s) > 0$, *then*

$$U_{k_1+k_2+\cdots+k_s+s-1}(A_1 A_2 \cdots A_s) \geqslant U_{k_1}(A_1)\, U_{k_2}(A_2) \cdots U_{k_s}(A_s).$$

For positive products this yields the following.

COROLLARY 3. *If* $k_1 \geqslant 0, k_2 \geqslant 0, \cdots, k_s \geqslant 0$ *with* $k_1 + k_2 + \cdots + k_s + s - 1 \geqslant n - 2$ *and* $U_{k_1}(A_1) > 0, U_{k_2}(A_2) > 0, \cdots, U_{k_s}(A_s) > 0$, *then if* $A_1 A_2 \cdots A_s = C = (c_{ij})$, *it follows that* $c_{ij} \geqslant U_{k_1}(A_1) U_{k_2}(A_2) \cdots U_{k_s}(A_s)$ *for each* $i \in N, j \in N$.

PROOF. Note that $U_{k_1+k_2+\cdots+k_s+s-1}(C) > 0$, i.e. $U_{n-2}(C) > 0$. Therefore by definition of $U_{n-2}$, $c_{ij} \geqslant U_{k_1}(A_1) U_{k_2}(A_2) \cdots U_{k_s}(A_s)$ for each $i \in N, j \in N$.

For a fixed measure of full indecomposability we have the following.

COROLLARY 4. *If* $U_k(A_i) > 0$ *for* $i = 1, 2, \cdots, s$, *for any* $s \geqslant [n, k]$, *where* $[n, k]$ *denotes the smallest integer larger than or equal to* $(n-1)/(k+1)$, *it follows that*

$$\prod_{i=1}^{s} A_i = C > 0 \quad \textit{and} \quad c_{ij} \geqslant \prod_{i=1}^{s} U_k(A_i), \quad \textit{for each } i \in N, j \in N.$$

PROOF. From Corollary 3 we need to find an integer $s$ so that $sk + s -$

$1 \geqslant n-2$. Solving for $s$ we have $s \geqslant (n-1)/(k+1)$. Hence $s \geqslant [n, k]$.

For fully indecomposable matrices this result translates into the following.

COROLLARY 5. *If each* $A_i$ *is fully indecomposable for* $i = 1, 2, \cdots, s$, *for any* $s \geqslant n-1$, *then* $\Pi_{i=1}^{s} A_i = C > 0$ *and* $c_{ij} \geqslant \Pi_{i=1}^{s} U_0(A_i)$, *for each* $i \in N$, $j \in N$.

All the above corollaries may be converted into results concerning $u_k$ by using Lemma 1 and the following.

LEMMA 6. *If* $D = \operatorname{diag}(d_1, d_2, \cdots, d_n)$ *is a diagonal matrix with positive main diagonal, then* $U_k(A+D) \geqslant \min\{d_1, d_2, \cdots, d_n, u_k(A)\}$.

PROOF. If $u_k(A) = 0$ the result is obvious. If $u_k(A) > 0$ then, by Lemma 3, pick $B$ minimal $u_k$ for which $b_{ij} \leqslant a_{ij}$, for all $i, j \in N$, and $u_k(B) = u_k(A)$. Since $B$ is minimal $u_k$, $b_{ii} = 0$ for all $i$, and $b_{ij} \geqslant u_k(A)$ for all $b_{ij} > 0, i \neq j$. As $u_k(A) > 0$, by Lemma 1, $U_k(B+D)$ is a positive entry of $B+D$ so that we must have

$$U_k(A+B) \geqslant U_k(B+D) \geqslant \min\{d_1, d_2, \cdots, d_n, u_k(A)\}.$$

Equality need not hold in this lemma as can be seen through the following example. Let

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \text{ and } D = \begin{pmatrix} 1/4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

Then $u_0(A) = 1$, and

$$U_0(A+D) = U_0 \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} = 1.$$

As a conversion example, Corollary 1 may be changed to read as follows.

CONVERTED COROLLARY 1. *Let* $D = \operatorname{diag}(d_1, d_2, \cdots, d_n)$ *and* $\mathcal{D} = (\delta_1, \delta_2, \cdots, \delta_n)$ *have positive main diagonals. If* $u_r(A) > 0$ *and* $u_s(B) > 0$, *then*

$$U_{r+s+1}[(A+D)(B+\mathcal{D})]$$
$$\geqslant \min\{d_1, d_2, \cdots, d_n, u_r(A)\} \cdot \min\{\delta_1, \delta_2, \cdots, \delta_n, u_s(B)\}.$$

**Application to bounds for eigenvalues and eigenvectors.** In this section we show that the results previously obtained lead to some interesting bounds for Perron vectors, the Perron root, and subdominate roots. In particular, we convert bounds concerning positive matrices into bounds concerning nonnegative matrices.

In [11] it is shown that if $B > 0$ and $x = (x_1, x_2, \cdots, x_n)^t$, a Perron vector of $B$, then $\max_{i,j}(x_i/x_j) \leqslant M/m$, where $M = \max_{i,j} b_{ij}$ and $m = \min_{i,j} b_{ij}$. Note also that if $U_k(A) > 0$, then $A$ is irreducible and, hence, has a positive Perron vector $x$.

THEOREM 1. *If* $U_k(A) > 0$, *and* $A$ *has Perron vector* $x = (x_1, x_2, \cdots, x_n)^t$ *then*

$$\max_{i,j} \frac{x_i}{x_j} \leqslant \frac{S^{[n,k]} - (n-1)[U_k(A)]^{[n,k]}}{[U_k(A)]^{[n,k]}},$$

*where* $S = \max_i \Sigma_k a_{ik}$.

PROOF. Let $S(A) = \max_i \Sigma_k a_{ik}$. Then it is easy to show that $S(A^m) \leqslant S(A)^m$. By Corollary 4, $A^{[n,k]} = C > 0$, with Perron vector $x$. As $\min_{i,j} c_{ij} \geqslant [U_k(A)]^{[n,k]}$, we have

$$\max_{i,j} c_{ij} \leqslant S^{[n,k]} - (n-1)[U_k(A)]^{[n,k]}.$$

The result is now immediate.

COROLLARY 6. *If* $u_k(A) > 0$ *with Perron vector* $x = (x_1, x_2, \cdots, x_n)^t$ *then*

$$\max_{i,j} \frac{x_i}{x_j} \leqslant \frac{[S + u_k(A)]^{[n,k]} - (n-1)[u_k(A)]^{[n,k]}}{[u_k(A)]^{[n,k]}}.$$

PROOF. Apply Theorem 1 to the matrix $A + u_k(A)I$ and use Lemma 6.

For bounds on the Perron root we add the following notations. Let

$$s = \min_i \left(\sum_k a_{ik}\right) \quad \text{and} \quad \min_{i,j} \frac{x_i}{x_j} \leqslant \delta,$$

where again $x = (x_1, x_2, \cdots, x_n)^t$ is a Perron vector of $A$.

THEOREM 2. *If* $U_k(A) > 0$ *then*

$$s^{[n,k]} + [U_k(A)]^{[n,k]}\left(\frac{1}{\delta} - 1\right) \leqslant [\rho(A)]^{[n,k]} \leqslant S^{[n,k]} - U_k(A)^{[n,k]}(1 - \delta).$$

PROOF. It is shown in [9, p. 153] that if $B > 0$, then $s + m(1/\delta - 1) \leqslant \rho(B) \leqslant S - m(1 - \delta)$, where, of course, the $s$, $m$, and $\delta$ refer to $B$. Now by

Corollary 4 we have for $A$,

$$s^{[n,k]} + [U_k(A)]^{[n,k]}\left(\frac{1}{\delta} - 1\right) \leqslant \rho(A)^{[n,k]}$$

$$\leqslant S^{[n,k]} - [U_k(A)]^{[n,k]}(1 - \delta).$$

Corollary 7. *If* $u_k(A) > 0$ *then*

$$[s + u_k(A)]^{[n,k]} + [u_k(A)]^{[n,k]}\left(\frac{1}{\delta} - 1\right) \leqslant [\rho(A) + u_k(A)]^{[n,k]}$$

$$\leqslant [S + u_k(A)]^{[n,k]} - [u_k(A)]^{[n,k]}(1 - \delta).$$

Proof. Apply Theorem 2 to the matrix $A + u_k(A)I$ and use Lemma 6.

Our initial results may also be used to determine bounds on subdominant roots, i.e. by the Perron-Frobenius theory if $A$ is primitive and $\mu$ is an eigenvalue of $A$, $|\mu| \neq \rho(A)$, then $|\mu| < \rho(A)$. We find bounds on such subdominant roots $\mu$.

Theorem 3. *If $A$ is generalized row stochastic, i.e., all row sums are equal, and $U_k(A) > 0$ then*

$$|\mu|^{[n,k]} \leqslant [\rho(A)]^{[n,k]} - [U_k(A)]^{[n,k]}n.$$

Proof. Lynn and Timlake [8, p. 147] show that if $B > 0$ and generalized row stochastic, then $|\mu| \leqslant \rho(B) - m \cdot n$. The result then follows, as in the previous theorem, from Corollary 4.

Corollary 8. *If $A$ is generalized row stochastic and $u_k(A) > 0$ then*

$$|\mu + u_k(A)|^{[n,k]} \leqslant [\rho(A) + u_k(A)]^{[n,k]} - [u_k(A)]^{[n,k]}n.$$

Proof. Apply Theorem 3 to the matrix $A + u_k(A)I$ and use Lemma 6.

**Application to Markov chain type problems and industrial production processes.** Our first result in this section concerns infinite products of nonnegative matrices. In particular we have the following.

Theorem 4. *Suppose that for each $A_i$ $(i = 1, 2, \cdots)$:*

(a) $\max_j r_j(A_i) \leqslant 1$ *where* $r_j(A) = \Sigma_{k=1}^n a_{jk}$ *for any matrix $A$.*

(b) $U_k(A_i) \geqslant \mu > 0$.

(c) *There is a number $\delta$, $0 < \delta < 1$, so that for each $i$ there is a $j(i)$ $\in N$ so that $r_{j(i)}(A_i) \leqslant \delta$.*

*Then* $\Pi_{i=1}^\infty A_i = 0$. *Further*

$$\max_j r_j \left[ \prod_{i=1}^{2[n,k]m} A_i \right] \leq [1 - \mu^{[n,k]}(1 - \delta)]^m$$

*for each positive integer* $m$.

PROOF. The proof is essentially given in [6].

A point of interest here is that $U_k$ is utilized in specifying a rate of convergence for $\Pi_{i=1}^{\infty} A_i$. Another result in [6] as follows.

THEOREM 5. *Suppose that each* $A_i$ $(i = 1, 2, \cdots)$ *is row stochastic with* $U_k(A_i) \geq \mu > 0$. *Then* $\Pi_{i=1}^{\infty} A_i = \lim_{N\to\infty}(A_N \cdots A_2 A_1) = A$ *where* $A$ *is row stochastic and rank one.*

This result may be converted into a meaningful result of Markov chain type as follows.

COROLLARY 9. *Suppose that each* $A_i$ $(i = 1, 2, \cdots)$ *is row stochastic with* $U_k(A_i) \geq \mu > 0$. *Further suppose that there is a vector* $y > 0$ *so that* $yA_i = y$ *for* $i = 1, 2, \cdots$. *Then* $\Pi_{i=1}^{\infty} A_i = \lim_{N\to\infty}(A_1 A_2 \cdots A_N)$ *is row stochastic and rank one.*

PROOF. As $yA_i = y$ for $i = 1, 2, \cdots$, there is a diagonal matrix $D = \operatorname{diag}(y_1, \cdots, y_n)$ so that $DA_iD^{-1}$ is column stochastic and, hence, $(DA_iD^{-1})^t = D^{-1}A_i^tD$ is row stochastic. Further there is some $\omega > 0$ so that $U_k(D^{-1}A_i^tD) \geq \omega$ for $i = 1, 2, \cdots$ and, hence, from Theorem 5,

$$A = \prod_{i=1}^{\infty} (D^{-1}A_i^tD) = \lim_{N\to\infty} (D^{-1}A_N^t \cdots A_2^tA_1^tD)$$

$$= \lim_{N\to\infty} (DA_1A_2 \cdots A_ND^{-1})^t,$$

where $A$ is rank one and row stochastic. Therefore

$$\lim_{N\to\infty} (DA_1A_2 \cdots A_ND^{-1}) = D\left[\lim_{N\to\infty} (A_1A_2 \cdots A_N)\right]D^{-1} = A^t$$

and so $\lim_{N\to\infty}(A_1A_2 \cdots A_N) = D^{-1}A^tD$. It is clear that $D^{-1}A^tD$ is rank one. Further $D^{-1}A^tD$ is row stochastic as each $A_1A_2 \cdots A_N$ is row stochastic and the row stochastic matrices form a closed set.

As a consequence of this corollary we have the following.

COROLLARY 10. *Suppose each* $A_i$ $(i = 1, 2, \cdots)$ *is doubly stochastic with* $U_k(A_i) \geq \mu > 0$. *Then*

$$\prod_{i=1}^{\infty} A_i = \lim_{N\to\infty} (A_1A_2 \cdots A_N) = A = \left(\frac{1}{n}\right).$$

PROOF. Note that $eA_i = e$ for $i = 1, 2, \cdots$ where $e = (1, 1, \cdots, 1)$. Hence $\lim_{N\to\infty}(A_1A_2 \cdots A_N) = A$ is rank one. As the set of doubly stochastic matrices forms a closed set and each $A_1A_2 \cdots A_N$ is doubly stochastic, it follows that $A$ is doubly stochastic. Finally we note that the only rank one doubly stochastic matrix is $(1/n)$.

The result may be interpreted in terms of Markov chain type problems as follows. Let $A_i$ represent the matrix of transitional probabilities of an $n$ state system at time $t_i$ for $i = 1, 2, \cdots$. If $A_1, A_2, \cdots$ satisfy Corollaries 9 or 10, then the system has final transitional probabilities given by $A$. Further results in this regard may be very beneficial.

Our next result extends a theorem of Richard Bellman [1], which has application to industrial production processes.

THEOREM 6. *Suppose for* $k = 1, 2, \cdots, n$ *we let* $A_k(q_k)$ *be a non-negative n-dimensional row vector depending on* $q_k \in S_k$, *some finite set. Let*

$$A(q) = \begin{pmatrix} A_1(q_1) \\ A_2(q_2) \\ \vdots \\ A_n(q_n) \end{pmatrix},$$

*where* $q = (q_1, q_2, \cdots, q_n) \in S = S_1 \times S_2 \times \cdots \times S_n$. *Consider the equation*

$$\text{(1)} \qquad \lambda y_i = \max_{q_i \in S_i} \sum_k a_{ik}(q_i) y_k, \qquad i = 1, 2, \cdots, n.$$

*If* $A(q)$ *has no zero columns for each* $q \in S$ *then* (1) *has a nonzero solution* $y \geq 0$. *If* $U_0[A(q)] > 0$ *for each* $q \in S$ *then* $y > 0$ *and is unique up to a multiplicative constant. Further, in this case,* $\lambda = \max_{q\in S}\rho[A(q)]$.

PROOF. We first show that a solution to (1) exists. For this consider the set $Y = \{y | y_i \geq 0$ and $\Sigma_i y_i = 1\}$ and the mapping of $Y$ into $R^n$ defined by

$$[M(y)]_i = \max_{q_i} \sum_k a_{ik}(q_i) y_k \Big/ \sum_j \max_{q_j} \sum_k a_{jk}(q_j) y_k.$$

Now

$$\sum_j \max_{q_j} \sum_k a_{jk}(q_j) y_k = \sum_j \sum_k a_{jk}(\overline{q}_j) y_k$$

for some $\overline{q} = (\overline{q}_1, \overline{q}_2, \cdots, \overline{q}_n)$ and so

$$\sum_j \sum_k a_{jk}(\bar{q}_j) y_k = \sum_k C_k[A(\bar{q})] y_k > 0$$

as $C_k[A(\bar{q})]$, the $k$th column sum of $A(\bar{q})$, is positive. Therefore $M$ is a continuous mapping from $Y$ into $Y$ and hence has a fixed point $\bar{y}$. Thus (1) has a nonzero solution.

Suppose now that $U_0[A(q)] > 0$ for each $q$. Then by Corollary 5, $A(w_1)A(w_2)\cdots A(w_{n-1}) = B(w_1, w_2, \cdots, w_{n-1}) > 0$ for all $(w_1, w_2, \cdots, w_{n-1}) \in S^{n-1}$. Consider the equation

$$\beta z_i = \max_{w \in S^{n-1}} \sum_k b_{ik}(w) z_k, \qquad i = 1, 2, \cdots, n. \tag{2}$$

By Bellman's result [1, p. 199], (2) has a unique positive solution $\beta$ and a solution $z > 0$ which is unique up to a multiplicative constant. For $B(\bar{q}, \bar{q}, \cdots, \bar{q}) = \Pi_{i=1}^{n-1} A(\bar{q})$, it is easily seen that

$$\lambda^{n-1} y_i = \max_{w \in S^{n-1}} \sum_k b_{ik}(w) y_k = \sum_k b_{ik}(\bar{q}, \bar{q}, \cdots, \bar{q}) y_k, \qquad i = 1, 2, \cdots, n.$$

By uniqueness then we have $\lambda^{n-1} = \beta$ and $y = cz$ for some scalar $c$. Further, by Bellman's result [1, p. 199],

$$\lambda^{n-1} = \max_w \rho[B(w)] = \rho[B(\bar{q}, \bar{q}, \cdots, \bar{q})] = \rho[A(\bar{q})]^{n-1}.$$

Therefore $\lambda = \max_q \rho[A(q)]$.

An interesting side result of this proof is as follows.

COROLLARY 11. *If*

$$A = \begin{pmatrix} A_1 \\ A_2 \\ \vdots \\ A_n \end{pmatrix} \geqslant 0, \quad B = \begin{pmatrix} B_1 \\ B_2 \\ \vdots \\ B_n \end{pmatrix} \geqslant 0 \quad \text{and} \quad C(q) = \begin{pmatrix} C(q_1) \\ C(q_2) \\ \vdots \\ C(q_n) \end{pmatrix}$$

*with* $C(q_k) = A_k$ *or* $B_k$, *then if* $U_0[C(q)] > 0$ *for each* $q$ *it follows that* $\rho(AB) \leqslant \max_q \rho[C(q)]^2$.

PROOF. By utilizing the latter part of the proof of Theorem 6, we may show that $\rho(AB)^{n/2} \leqslant \max_q \rho[C(q)]^n$ if $n$ is even, or $\rho(AB)^{(n+1)/2} \leqslant \max_q \rho[C(q)]^{n+1}$ if $n$ is odd. Thus $\rho(AB) \leqslant \max_q \rho[C(q)]^2$.

For example if

$$A = \begin{pmatrix} 1/3 & 2/3 \\ 1/3 & 2/3 \end{pmatrix} \quad \text{and} \quad B = \frac{1}{4}\begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix},$$

then $\rho(A) = 1$, $\rho(B) = 1$, yet $\rho(AB) = 13/12$. Note however that

$$\rho\begin{pmatrix} 1/3 & 2/3 \\ 3/4 & 1/2 \end{pmatrix} = \frac{5+\sqrt{73}}{12}$$

and that

$$13/12 < ((5+\sqrt{73})/12)^2.$$

This seems an interesting inequality in itself as there are few known bounds on $\rho(AB)$.

**Summary.** In summary, our initial results lay down the combinatorial principles of the concepts $U_k$ and $u_k$. This is followed by our application sections indicating the usefulness of $U_k$ and $u_k$ in converting positive matrix results into nonnegative matrix results. Conversions other than those presented herein are no doubt also possible.

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